STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING ZUIVERE WISKUNDE

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Topological semigroups

by

Aïda Paalman - de Miranda

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CHAPTER I

§ 1. Subgroups and subsemigroups

Definition 1: A topological semigroup ("mob") is a space S together with a continuous function $f : S \times S \to S$, such that

a) S is a Hausdorff space

b) f is associative.

If we write $f(x,y) = xy$, then (b) becomes the more familiar $(xy)z = x(yz)$. A mob may be thought of as a set of elements which is both an abstract semigroup and a Hausdorff space, the operation of the semigroup being continuous in the topology of the space.

Familiar examples are the topological groups and the closed unit interval with the usual multiplication.

In all that follows S will be a mob.

Definition 2: A subsemigroup of S is a non-void set $A \subset S$ satisfying $A^2 \subset A$, with $A^2 = \{ xy \mid x,y \in A \}$.

Definition 3: A subgroup of S is a non-void set $A \subset S$, satisfying $xA = Ax = A$, for all $x \in A$.

Of course definition 3 defines an abstract group in the customary sense. A however, with the relative topology, need not be a topological group, since the function f; with $f(x) = x^{-1}$ $(x, x^{-1} \in A)$ need not be continuous.

Lemma 1: Let A be a subsemigroup of S. Then $\overline{A}$ is a subsemigroup of S.

Proof: Suppose for $x,y \in \overline{A}$, $xy \notin \overline{A}$. Then since $\overline{A}$ is closed, there exists neighbourhoods V of x and W of y, such that $V.W \cap \overline{A} = \emptyset$.

Since $x,y \in \overline{A}$, there is an $a_1 \in V \cap A$ and $a_2 \in W \cap A$. This implies $a_1 a_2 \in V.W$ and $a_1 a_2 \notin \overline{A}$, which is a contradiction.

Theorem 1. Each subgroup of S is contained in a maximal subgroup, and no two of these intersect.

Proof. Let A be a subgroup of S and e the identity of A. Let A_o be the set of all $a \in S$, such that $ae = ea = a$, and such that there exists an $a^{-1} \in S$, with $aa^{-1} = a^{-1}a = e$, $a^{-1}e = a^{-1}$. Then it is immediately clear that A_o is a maximal subgroup of S containing A.

Now suppose A_1 and A_2 are maximal subgroups of S, $a \in A_1 \cap A_2 \neq \emptyset$.
Let e_1 and e_2 be the identities of A_1 and A_2, and let $aa_1^{-1} = e_1$, $aa_2^{-1} = e_2$.
Then $a_2^{-1}a\, e_1 = a_2^{-1}a = e_2e_1 = e_2$

$$e_2aa_1^{-1} = aa_1^{-1} = e_2e_1 = e_1.$$

Hence $e_1 = e_2$.
Since A_1 is maximal, A_1 contains all a with $ae_1 = e_1a = a$ and $aa^{-1}=a^{-1}a=e$.
Hence $A_1 = A_2$.

It may happen that S contains no subgroups. Consider for example the open unit interval I = (0,1), with the usual multiplication. I contains no subgroups.

<u>Lemma 2.</u> Let S be a mob and B^* a compact subset of S.
$A = \{a_\lambda\}_{\lambda \in \Lambda}$, $B = \{b_\lambda\}_{\lambda \in \Lambda}$ $A \subset S$, $B \subset B^*$.
Then to every $a \in \bar{A}$ there exists a $b \in \bar{B}$, such that $ab \in \bar{C}$, with
$\bar{C} = \overline{\{a_\lambda b_\lambda\}}_{\lambda \in \Lambda}$

<u>Proof.</u> Let $\{V_\tau\}_\tau$ be a complete system of neighbourhoods of a.
$A_\tau = V_\tau \cap A \neq \emptyset$ and $B_\tau = \{b_\alpha \mid a_\alpha \in A_\tau\}$. $\{B_\tau\}_\tau$ is a family of subsets of B^* with the finite intersection property. For let $B_{\tau_1}, \ldots, B_{\tau_n}$ be any finite number of sets. Since $V_{\tau_1}, \ldots, V_{\tau_n}$ are neighbourhoods of a and $\{V_\tau\}_\tau$ is a complete system of neighbourhoods, there exists a $V_{\tau_o} \in \{V_\tau\}_\tau$, such that $V_{\tau_o} \subset \bigcap_{i=1}^{n} V_{\tau_i}$. Since $A_{\tau_o} \neq \emptyset$, we have $B_{\tau_o} \neq \emptyset$, $B_{\tau_o} \subset \bigcap_{i=1}^{n} B_{\tau_i}$. Hence by the compactness of B^*, $\bigcap_\tau \bar{B}_\tau \neq \emptyset$.
Let $b \in \bigcap_\tau \bar{B}_\tau$ and let V be any neighbourhood of ab. Then there exist neighbourhoods V_α of a and W of b, such that $V_\alpha.W \subset V$.
Since $b \in \bigcap_\tau \bar{B}_i$, $W \cap B_\alpha \neq \emptyset$. Let $b_{\alpha_o} \in W \cap B_\alpha$, then $a_{\alpha_o} \in A_\alpha = A \cap V_\alpha$.
Hence $a_{\alpha_o} b_{\alpha_o} \in V_\alpha.W \subset V$.
On the other hand $a_{\alpha_o} b_{\alpha_o} \in C$, and thus $ab \in \bar{C}$.

<u>Theorem 2.</u> If S is compact, then each maximal subgroup of S is closed.
<u>Proof.</u> Let A be a maximal subgroup of S and suppose $A \not\subset x\bar{A}$ for $x \in \bar{A}$.
Then there is an $a \in A$, with $a \neq xa_\nu$ $a_\nu \in \bar{A}$.
Hence there exists neighbourhoods $V(a_\nu)$ and $W_\nu(x)$ with $a \notin W_\nu(x).V_\nu(a_\nu)$.

Since A is compact, $\overline{A}$ is covered by a finite number of $V_\nu(a_\nu)$'s, say say $V_1(a_1)\ldots,V_n(a_n)$. Let $W(x) = \bigcap_{i=1}^{n} W_i(x)$. Then $a \notin W(x)\overline{A}$.
Since $x \in \overline{A}$, there is a $b \in A \cap W(x)$; and hence $a \notin b\overline{A}$, which is a contradiction. Thus $A \subset x\overline{A}$ for all $x \in \overline{A}$, and hence $\overline{A} \subset x\overline{A}$ for all $x \in \overline{A}$.
Analoguously $\overline{A} \subset \overline{A}x$.
Since $Aa = aA = A$ for all $a \in A$, we have by lemma 2 $a\overline{A} \subset \overline{A}$, $\overline{A}a \subset \overline{A} \Rightarrow$ $\overline{A}A \subset \overline{A}$ and $A\overline{A} \subset \overline{A}$.
Since $Ax \subset \overline{A}$ and $xA \subset \overline{A}$ for all $x \in \overline{A}$, we have again by lemma 2 $\overline{A}x \subset \overline{A}$, $x\overline{A} \subset \overline{A}$. Hence $\overline{A}x = x\overline{A} = \overline{A}$ for all $x \in \overline{A}$, and $\overline{A}$ is a subgroup of S.
Since A is maximal, we have $A = \overline{A}$.

Let I be the semigroup $[0,\infty)$ with the usual multiplication. $A = (0,\infty)$ is a subgroup of I. $\overline{A}$, however, is no subgroup of I.

Lemma 3. Let S be a locally compact mob and an abstract group. Let A be a countable subset of S, and $x \in \overline{A}$. Then $x^{-1} \in \overline{A^{-1}}$.

Proof. Let $E = A \cup \{x\}$ and $B = \bigcup_{n=-\infty}^{\infty} E^n$. Then B is a countable subgroup of S and the continuity of multiplication implies $\overline{B}^2 \subset \overline{B}$.

Now let V be a compact neighbourhood of the identity. Then since S is a group we get $\overline{B} \subset BV^{-1}$.
Thus $\overline{B} = \bigcup_{b \in B} [bV^{-1} \cap \overline{B}] = \bigcup_{b \in B} [b(V^{-1} \cap \overline{B})]$.

By lemma 2 V^{-1} is closed, since V is compact and hence $b(V^{-1} \cap \overline{B})$ is closed. Moreover $\overline{B}$ is a closed subset of S and hence locally compact. Baires category theorem implies that the interior relative to $\overline{B}$ of one of the sets $b(V^{-1} \cap \overline{B})$ is not empty. Hence there exists an open set $U \neq \emptyset$, and an element $b_o \in B$, such that $U \cap \overline{B} \subset b_o(V^{-1} \cap \overline{B})$.
Let $c \in B \cap U$, then
$xc^{-1}(U \cap \overline{B}) = xc^{-1}U \cap \overline{B}$. $xc^{-1}U = U_o$ is open.
$U_o \cap E \subset U_o \cap \overline{B} \subset xc^{-1}(U \cap \overline{B}) \subset xc^{-1}b_oV^{-1}$.
$(U_o \cap E)^{-1} \subset Vb_o^{-1}cx^{-1} = C$. C is compact..

Then by lemma 2, there exists to every $a \in \overline{U_o \cap A}$ an element $b \in \overline{(U_o \cap A)^{-1}}$, with ab the identity. Hence since $x \in \overline{U_o \cap A}$,
$x^{-1} \in \overline{(U_o \cap A)^{-1}} \subset \overline{A^{-1}}$.

Lemma 4. Let S be a locally compact mob and an abstract group. Let A be a compact subset of S. Then A^{-1} is compact.

Proof. By lemma 2 A^{-1} is closed.

Suppose A^{-1} cannot be covered by a finite number of compact subsets $x_i^{-1}V$, with V any compact neighbourhood of the identity, $x_i \in A$.
Then there is a sequence $\{x_n^{-1}\}_{n=1}^{\infty} \subset A^{-1}$, such that $x_n^{-1} \notin \bigcup_{i=1}^{n-1} x_i^{-1}V$. Let $E_n = \{x_k \mid k \geq n\}$. Since A is compact, there exists an $y \in \bigcap_{n=1}^{\infty} \overline{E}_n$.
Since $y \in \overline{E}_1$, there is $x_m \in Vy$, whence $y^{-1} \in x_m^{-1}V$. Moreover $y \in \overline{E}_{m+1}$ implies by lemma 3 $y^{-1} \in \overline{E_{m+1}^{-1}}$. Thus there is an $n > m$ such that $x_n^{-1} \in x_m^{-1}V$, which contradicts the choice of $\{x_n^{-1}\}$.

Theorem 3. Let S be a locally compact mob and an abstract group. Then S is a topological group.

Proof. Let U be an open neighbourhood of the identity u, and $\{V_\alpha\}_\alpha$ the collection of compact neighbourhoods of u. Suppose that for every V_α, $V_\alpha^{-1} \not\subset U$. Then $V_\alpha^{-1} \cap C(U) \neq \emptyset$.
$\bigcap_\alpha V_\alpha^{-1} \cap C(U) \neq \emptyset$, since V_α^{-1} is compact.
But $\bigcap_\alpha V_\alpha^{-1} \cap C(U) \subset \bigcap_\alpha V_\alpha^{-1} = u$ implies that $u \in C(U)$, which is a contradiction. Hence to every neighbourhood U of u there exists a neighbourhood V of u, such that $V^{-1} \subset U$.
This implies that S is a topological group.

Let S be the additive group of real numbers. We define a topology in S by means of a base B consisting of all half open intervals $[a,b)$. S is a mob and an abstract group. S however is no topological group, for there is no neighbourhood U of 1, with $-U \in [-1,-\frac{1}{2})$.

An element e of a mob is called an idempotent if $e^2 = e$. We shall denote by E the set of idempotents in S.
If S contains an idempotent e, then $\{e\}$ is a subgroup of S, and is contained in a maximal subgroup.
By H(e) we shall denote the maximal subgroup of S containing the idempotent e.

An element O is termed the zero of S if $Ox = xO = O$ for all x en S. Then it is easily seen that the zero of S if it exists is uniquely defined. It is also immediately clear that it is an idempotent.

An element u is termed the identity of S if $ux = xu = u$ for all $x \in S$. Then the identity of S, if it exists is uniquely defined and is an idempotent.

A semigroup S in which the product of any two elements of S is the zero of S, we term a zero semigroup.

<u>Lemma 5.</u> The set E of all idempotents of a topological semigroup is closed.

<u>Proof.</u> If $E = \emptyset$. Lemma 5 is trivial.

Suppose that $p \in \overline{E}$. If $p^2 \neq p$, then there exists a neighbourhood V of p such that $V^2 \cap V = \emptyset$.

Since $p \in \overline{E}$, there is an $e \in E$, with $e \in V$, and hence $e = e^2 \in V^2 \cap V$, which is a contradiction.

<u>Theorem 4.</u> Let S be a compact mob. Then S contains a subgroup, and hence at least one idempotent element.

<u>Proof.</u> Let $a \in S$ and $A_n = \{ a^i \mid i \geqslant n \}$.

Then since S is compact $D = \bigcap_{n=1}^{\infty} \overline{A}_n \neq \emptyset$.

D is a subsemigroup of S, and the continuity of the multiplication implies that D is commutative.

Suppose now $xD \neq D$, $x \in D$.

Then there exists $z \in D$, such that $z \neq xd_\lambda$ for all $d_\lambda \in D$.

Therefore there exist neighbourhoods $V_\lambda(x)$, $V_\lambda(d_\lambda)$ and $V_\lambda(z)$ such that $V_\lambda(x)\ V_\lambda(d\ ) \cap V_\lambda(z) = \emptyset$.

Since D is compact we can choose a finite sub covering $V_{\lambda_1}(d_{\lambda_1}), \ldots, V_{\lambda_n}(d_{\lambda_n})$.

Let $V(x) = \bigcap_{i=1}^{n} V_{\lambda_i}(x)$, $V(z) = \bigcap_{i=1}^{n} V_{\lambda_i}(z)$ and $\mathcal{O} = \bigcup_{i=1}^{n} V_{\lambda_i}(d_{\lambda_i})$.

Then $V(x) . \mathcal{O} \cap V(z) = \emptyset$..

Since $x, z \in D$ there is $a^m \in V(x)$ and $a^{n_i} \in V(z)$ with $n_{i+1} > n_i > m$ $(i=1,2,\ldots)$.

Set $k_i = n_i - m$.

Then $\bigcap_{i=1}^{\infty} \overline{A}_{k_i} \neq \emptyset$. Let $b \in \bigcap_{i=1}^{\infty} \overline{A}_{k_i} \subset D$.

Choose $V(b) \subset \mathcal{O}$ and $a^{k_j} \in V(b)$.

Then $a^{n_j} \in V(z)$ and $a^{n_j} = a^{k_j+m} = a^m . a^{k_j} \in V(x).V(b) \subset V(x).\mathcal{O}$, which is a contradiction.

Hence xD = Dx = D, and D is a subgroup of S.

<u>Corollary</u>: Let S be a topological semigroup and S' a subsemigroup that is compact. Then if S is a group, S' is a subgroup.

<u>Proof.</u> By theorem 4, S' contains an idempotent which must be u (the identity of S).

Again by theorem 4, applied to xS', $x \in S'$, there is an idempotent in xS'. Thus $u \in xS'$ and $S' = uS' \subset xS'$.

Hence since $xS' \subset S'$, $xS' = S'$ for all $x \in S'$. Analogously $S'x = S'$.

Let S be a mob and let $H = \{ y \mid y \cup Sy = x \cup Sx \text{ and } y \cup yS = x \cup xS \}$.

Then $H_e = H(e)$ for $e \in E$.

For let $x \in H(e)$, then $x = ex = xe$, and hence $x \cup Sx \subset e \cup Se$, $x \cup xS \subset e \cup eS$. Since $e = x^{-1}x = xx^{-1}$, we have $e \cup Se \subset x \cup Sx$ and $e \cup eS \subset x \cup xS$. Thus $H(e) \subset H_e$.

Let now $x \in H_e$. Then since $x \cup Sx = e \cup Se$, and $x \cup xS = e \cup eS$, $xe = ex = e$, and x has a left and right inverse, hence $x \in H(e)$. This implies $H(e) = H_e$.

<u>Lemma 6.</u> If S is compact, then $\mathcal{H} = \{ (x,y) \mid x,y \in H_x = H_y \}$ is a compact subset of $S \times S$.

<u>Proof.</u> Let $H \neq S \times S$ and let $(x,y) \in S \times S \setminus H$. Then we may assume $x \notin Sy$ (or $x \notin yS$, or $y \notin xS$, or $y \notin Sx$).

If $x \notin Sy$, $Sy \subset S \setminus x$ and hence $Sy \subset S \setminus \overline{V}$ for some open set V about x, since S is regular and Sy is closed. Again from the compactness of S we can find an open set U about y such that $SU \subset S \setminus \overline{V}$. Hence $(V \times U) \cap \mathcal{H} = \emptyset$ and we may infer that H is closed.

<u>Theorem 5.</u> If S is compact, then $H = \bigcup \{ H(e) \mid e \in E \}$ is closed. If $x \in H$, let $\alpha(x)$ be the unit of the unique maximal subgroup containing x and let $\beta(x)$ be the inverse of x in this group.

Then $\alpha : H \to E$ is a retraction and $\beta : H \to H$ is a homeomorphism.

<u>Proof.</u> Let $p : S \times S \to S$, $p(x,y) = x$.
Then $H = \bigcup \{ H(e) \mid e \in E \} = p(\mathcal{H} \cap S \times E)$.
Hence H is closed since $\mathcal{H}$ and E are closed.
Let $A = \{ (x, \beta(x) \mid x \in H \}$ and $m : S \times S \to S$, $m(x,y) = xy$.
Then $A = \mathcal{H} \cap H \times H \cap m^{-1}(E)$.
For let $(x, \beta(x)) \in A$, then $m(x, \beta(x)) \in E$, furthermore $x, \beta(x) \in H(e) \Rightarrow (x, \beta(x)) \in H \times H$ and $(x, \beta(x)) \in \mathcal{H}$.

If $(x,y) \in \mathcal{H} \cap H \times H \cap m^{-1}(E)$ then $xy = e \in E$.
Since $(x,y) \in \mathcal{H} \Rightarrow H_x = H_y$ and $x,y \in H$ implies $H_x = H_y = H(e_1) = H(e)$.
Hence $y = \beta(x)$.
Since $\mathcal{H}$, $H \times H$ and $m^{-1}(E)$ are closed, A is compact and since $p|A$ is continuous p/A is topological. Hence $p^{-1}\ x \to (x, \beta(x))$ is continuous.
Thus β is continuous.
α is continuous since $\alpha(x) = x\beta(x)$.

§ 2. Ideals

Definition 1. A non-empty subset A of S is called a left ideal if $SA \subset A$, a right ideal if $AS \subset A$, and an ideal if it is both a left and a right ideal.

A minimal left (right) ideal of S is a left (right) ideal containing no other left (right) ideal.
We shall denote by $\mathcal{L}(S)$ and $\mathcal{R}(S)$ respectively the collections of all minimal left and all minimal right ideals of S.
In general these may be empty collections. The intersection of all ideals of S is called the kernel of S and denoted by K.
If K is non-empty it is clearly the smallest ideal of S.

Lemma 1. Let A be an ideal of S. Then $\bar{A}$ is an ideal of S.

Proof. Since $SA \subset A$ and $AS \subset A$ the continuity of multiplication implies $S\bar{A} \subset \bar{A}$ and $\bar{A}S \subset \bar{A}$. Hence $\bar{A}$ is an ideal of S.

If $a \in S$ we let

$$J(a) = \{a\} \cup Sa \cup aS \cup SaS$$
$$L(a) = \{a\} \cup Sa$$
$$R(a) = \{a\} \cup aS.$$

Thus J(a) is the smallest ideal of S which contains a. L(a) and R(a) are respectively the smallest left and right ideal of S which contain a.

If $A \subset S$ then we define $J_o(A)$ to be the null-set if A contains no ideal of S and $J_o(A)$ is the union of all ideals contained in A in the contrary case. $L_o(A)$ $(R_o(A))$ is the null-set if A contains no left right ideal of S and $L_o(A)$ $(R_o(A))$ is the union of all left (right) ideals contained in A in the contrary case.
It is clear that if $J_o(A) \neq \emptyset$, then $J_o(A)$ is the largest ideal of S contained in A.
Also if $L_o(A) \neq \emptyset$ and $R_o(A) \neq \emptyset$, then $L_o(A)$ is the largest left and $R_o(A)$ is the largest right ideal of S contained in A.

Lemma 2. If $A \subset S$ is closed, then $J_o(A)$ is closed. If A is open and if S is compact, then $J_o(A)$ is open.

Proof. $J_o(A)$ is the largest ideal of S contained in A. Since $\overline{J_o(A)} \subset \bar{A}$ and $\overline{J_o(A)}$ is an ideal of S by lemma 1, $\overline{J_o(A)} \subset J_o(A)$ if $A = \bar{A}$.

Hence $J_o(A)$ is closed if A is closed.

Suppose now that S is compact and A is open.
Let $x \in J_o(A)$; then $\{x\} \cup xS \cup Sx \cup SxS = J(x) \subset A$. Since A is open and S is compact there exists an open set V about x satisfying

$$V \cup VS \cup SV \cup SVS \subset A.$$

Now this set is an ideal of S, hence is contained in $J_o(A)$. Therefore $x \in V \subset J_o(A)$ completing the proof.

<u>Theorem 1.</u> Let S be compact; then any proper ideal of S is contained in a maximal proper ideal of S, and each maximal proper ideal is open.

<u>Proof.</u> If the ideal $I \neq S$, then lemma 2 shows that $J_o(S \setminus x)$ is an open proper ideal for any $x \in S \setminus I$. Let $\{T_\alpha\}_\alpha$ be a linearly ordered system of open proper ideals containing I.
If $S = \bigcup_\alpha T_\alpha = T$, then S is the union of a finite number of T_α's because S is compact. Since $\{T_\alpha\}$ is linearly ordered, there is a α with $S = T_\alpha$, which is a contradiction. Hence $T = \bigcup_\alpha T_\alpha$ is a proper ideal of S.
Using Zorn's lemma there is a maximal element in the collection of all open proper ideals containing I.
Each maximal proper ideal M is open, since $M \subset J_o(S \setminus x)$, $x \notin M$.

<u>Remark.</u> An analoguous result holds for left and right ideals. Thus if S is compact, then any proper left (right) ideal of S is contained in a maximal proper left (right) ideal of S, and each maximal proper left (right) ideal is open.

<u>Corollary</u>. If S is compact and connected and J is a maximal proper ideal of S, then J is dense in S.

<u>Proof.</u> Since J is open, and $\overline{J}$ an ideal of S, the maximality of J implies $\overline{J} = S$.

Let S be the multiplicative semigroup of real numbers, with the usual topology.
Then zero is the only proper ideal of S. Hence zero is a maximal proper ideal which is not open. Also if $A = (-1,1)$, then $J_o(A) = 0$.

<u>Lemma 3.</u> If S is compact, then J(a) is compact. The same holds for L(a) and R(a).

Proof. $J(a) = \{a\} \cup Sa \cup aS \cup SaS$.

Since S is compact, $\{a\}$, Sa, aS and SaS are compact subsets of S. Thus J(a) is compact.

Theorem 2. If S has a minimal left and a minimal right ideal, then S has a minimal ideal K, and

1^o. If A_1 and A_2 are both in $\mathcal{L}(S)$ or both in $\mathcal{R}(S)$ and $A_1 \cap A_2 \neq \emptyset$, then $A_1 = A_2$.

2^o. If $L \in \mathcal{L}(S)$ then $La = L$ for all $a \in L$. If $R \in \mathcal{R}(S)$ then $aR = R$ for all $a \in R$.

3^o. $K = \bigcup_{L \in \mathcal{L}(S)} L = \bigcup_{R \in \mathcal{R}(S)} R$.

Proof. 1) If A_1 and A_2 are in $\mathcal{L}(S)$, then $A_1 \cap A_2$ is a left ideal of S and thus $A_1 = A_1 \cap A_2 = A_2$.

2) If $a \in L$, La is a left ideal contained in L, hence $La = L$.
The same argument works for right ideals.

3) If $L \in \mathcal{L}(S)$ and $a \in S$, then La is a left ideal of S and $La \in \mathcal{L}(S)$. For if L_o were a left ideal properly contained in La, then $L \cap \{x \mid xa \in L_o\}$ would be a left ideal properly contained in L.

Thus $\bigcup_{a \in S} La$ is a union of left ideals in $\mathcal{L}(S)$ and is an ideal.
If I is any ideal of S and $L_1 \in \mathcal{L}(S)$, then $L_1 = IL_1 \subset I$, so I contains $\bigcup_{a \in S} L_1 a$, which must by definition be the kernel K. Also any $L_2 \in \mathcal{L}(S)$ must be contained in K, since K is an ideal.
So by 1) L_2 must be equal to $L_1 a$ for some $a \in S$. Thus $K = \bigcup_{L \in \mathcal{L}(S)} L$.

The same argument applies to right ideals.

Let S be the semigroup (0,1) with the usual multiplication.
The kernel K of S is empty, since for any $a \in S$ $S(0,a) = (0,a)S \subset (0,a)$.

Theorem 3. If S satisfies the conditions of theorem 2, then
1^o) If $L \in \mathcal{L}(S)$ and $R \in \mathcal{R}(S)$, then $L \cap R$ is a subgroup of S.
2^o) $\mathcal{L}(S) = \{S(e) \mid e \in K \cap E\}$. $\mathcal{R}(S) = \{eS \mid e \in K \cap E\}$.
3^o) $K = \bigcup \{H(e) \mid e \in E \cap K\}$.

Any pair $H(e_1)$, $H(e_2)$ of subgroups with $e_1, e_2 \in E \cap K$ are topologically isomorphic.

Proof. Choose $L \in \mathcal{L}(S)$ and $R \in \mathcal{R}(S)$.

1) Then $RL \subset L \cap R$, so $L \cap R \neq \emptyset$.

If $a \in L$ and $b \in R$, then

$$(L \cap R)a = L \cap R \text{ and } b(L \cap R) = L \cap R.$$

For it is clear that $(L \cap R)a \subset L \cap R$, and if the inclusion were proper then

$$La = \bigcup_{R \in \mathcal{R}(S)} (L \cap R)a \subset \bigcup_{R \in \mathcal{R}(S)} (L \cap R) = L = La \quad \text{is a contradiction.}$$

The equality $b(L \cap R) = L \cap R$ follows similarly. $L \cap R$ is a subgroup of S since for all $a \in L \cap R$ $\quad (L \cap R)a = a(L \cap R) = L \cap R$.

2) Let e be the identity element of $L \cap R$, then clearly Se is a left ideal contained in L, hence $L = Se$ and $R = eS$.

$L \cap R = Se \cap eS \supset eSe = eL \supset e(L \cap R) = L \cap R$.

Hence $L \cap R = eSe$.

3) $L \cap R$ is the maximal subgroup containing e, for $H(e) = e.H(e).e \subset eSe$. Hence $H(e) = eSe = L \cap R$. By theorem 4 we have the disjoint union

$$K = \bigcup_{L \in \mathcal{L}(S)} \quad L = \bigcup_{R \in \mathcal{R}(S)} \quad R = \bigcup_{L,R} L \cap R.$$

Thus $K = \bigcup_{e} \{ H(e) \mid e \in E \cap K \}$.

We shall now prove that any pair $H(e_1)$, $H(e_2)$ with $e_1, e_2 \in E \cap K$ are topologically isomorphic. It is clear that if $H(e_1) \subset L$ and $H(e_2) \subset L$, then $Le_1 = Le_2 = L$. Let $\varphi : H(e_1) \longrightarrow L$ be given with $\varphi(x) = e_2 x$.

Then $e_2 x \in H(f)$, $f \in L \cap E$.

Let $\bar{x}$ be the inverse of $e_2 x$ in $H(f)$. Thus $e_2 x \bar{x} = \bar{x} e_2 x = f$.

And so $e_2 f = e_2^2 x \bar{x} = f$, hence $f = e_2$.

It is clear then that φ is a map of $H(e_1)$ onto $H(e_2)$. We easily verify that φ is a homomorphism.

If $e_2 x = e_2 y$, then $e_1 e_2 x = e_1 e_2 y \Longrightarrow e_1 x = e_1 y \Longrightarrow x = y$.

Hence φ is an isomorphism.

Since $\varphi^{-1}(x) = e_1 x \quad x \in H(e_2)$, φ and φ^{-1} are both continuous.

Hence $H(e_1)$ and $H(e_2)$ are topologically isomorphic. In the same way $H(e_1)$ and $H(e_2)$ both in R implies $H(e_1)$ and $H(e_2)$ topologically isomorphic.

Suppose now $H(e_1) = L_1 \cap R_1$ and $H(e_2) = L_2 \cap R_2$, then $H(e_1) \simeq L_1 \cap R_2 \simeq H(e_2)$.

Theorem 4. Let S be a compact mob. Then each left ideal of S contains at least one minimal left ideal of S and each minimal left ideal is closed. The same holds for right ideals.

Proof. Let I be any left ideal of S and let Q be the collection of all closed left ideals of S contained in I. Q is partially ordered by inclusion and is non-void, since if $x \in I$, Sx is a closed left ideal contained in I. Suppose Q' is a linearly ordered sub collection of Q. Then $\bigcap_{A \in Q'} A$ is non-empty since S is compact, and so is an ideal in Q.
Thus Q' has a lower bound and Zorn's lemma assures the existence of a minimal L_o in Q.
Let L_1 be a left ideal contained in L_o and let $x \in L_1$. Then Sx is a closed left ideal.
Furthermore $Sx \subset L_1 \subset L_o$ and since L_o is minimal in Q $L_o = Sx$, hence $L_1 = L_o$. Thus L_o is a minimal left ideal.
The proof of the assertion for right ideals is completely analogous.

Corollary 1. Let S be a compact mob. Then S has a minimal ideal K.

Corollary 2. Let S be a commutative compact mob. Then K is a compact topological group.

Proof. If S is commutative and J_1 and J_2 are minimal ideals, then $J_1 \cap J_2$ is non empty since it contains $J_1 J_2$.
Thus $J_1 = J_2$ and $J_1 \cap J_2 = J_1$ is a subgroup of S. Since $K = J_1$, K is a subgroup of S.

Lemma 4. Let S satisfy the conditions of theorem 2. Then
$K = \{ Se \cap E \} . eSe. \{ eS \cap E \}$ $e \in E \cap K$.

Proof. Suppose $e \in E \cap K$, then by theorem 3, Se is a minimal left ideal and eS is a minimal right ideal. Furthermore $H(e) = e.H(e) \subset eKe \subset eSe = H(e)$. Hence $H(e) = eSe = eKe$.
Let $H(e_1) = L \cap R$, $H(f) = Se \cap R$, $H(g) = L \cap eS$, $f \in Se$, $g \in eS$. Then by theorem 3 $H(f) = fH(e)$. $H(e_1) = fH(e)e_1 = f\,e\,Se\,e_1$. Analogously $H(g) = H(e).g$ and $H(e_1) = e_1 H(e) g = e_1 eSe\,g$. This implies that

$$H(e_1) = H(e_1).H(e_1) = feSee_1e_1eSeg = feSxSeg.$$

SxS is an ideal of S, and since $x \in K$, $SxS \subset K$, which implies $SxS = K$.

Thus $H(e_1) = feKeg = feSeg$.

Thus $K = \bigcup_e \{H(e) \mid e \in E \cap K\} = \{Se \cap E\}.eSe.\{eS \cap E\}$. $e \in E \cap k$.

Theorem 5. If S is compact and if K is the minimal ideal of S then K is a retract of S.

Proof. Define $f : S \to K$ by $f(x) = \alpha(xe).exe.\ \alpha(ex)$, $e \in E \cap K$ where $\alpha(xe)$ is the unit of the unique maximal subgroup $\subset Se$ containing xe.

§ 1. theorem 5 implies that $g : x \to \alpha(xe)$ is a continuous mapping of S into E. Hence f is continuous. Let $x \in K$, then by Lemma 4 $x = e_2.eye.e_3$, with $e_2 \in Se$ and $e_3 \in eS$.

Hence $ee_2 = e$ and $e_3e = e$.

Furthermore $exe = ee_2eyee_3e = e^2ye^2 = eye$.

$xe = e_2eyee_3e = e_2eye \in e_2H(e) = H(e_2)$. Hence $\alpha(xe) = e_2$.

$ex = ee_2eyee_3 = eyee_3 \in H(e)e_3 = H(e_3)$. Hence $\alpha(ex) = e_3$.

Thus if $x \in K$, then $f(x) = \alpha(xe)exe.\ \alpha(ex) = e_2eyee_3 = x$, and hence f is a retraction of S on K.

Theorem 6. Let S be a compact mob and let $e \in E$. Then these are equivalent.

i) Se is a minimal left ideal

ii) SeS is the minimal ideal

iii) eSe is a maximal subgroup.

Proof. (i) $\to$ (ii). Let I be an ideal of S.

Since $L = Se$ is a minimal left ideal, $IL = L \subset I$.

Hence L is contained in every ideal of S. Thus $LS \subset I$ and since $LS = SeS$ is an ideal of S, $SeS = K$.

(ii) $\to$ (iii) If $SeS = K$, then $e \in K$, and hence theorem 3 implies that Se is a minimal left and eS is a minimal right ideal.

Hence $eS \cap Se = eSe$ is a maximal subgroup.

(iii) $\to$ (i), Let L be a left ideal contained in Se, and let $a \in L \cap eS$, Then since $a \in Se \cap eS = eSe$, there is an element $a^{-1} \in eSe$ such that $a^{-1}a = e$. Hence $e = a^{-1}a \in a^{-1}L \subset L$. Thus $L = Se$.

Remark. If S contains a zero element O, then theorem 2,3,4 and 5 become trivial, since then O is the minimal (left,right) ideal of S.

§ 3. Simple semigroups

If an ideal I of a mob S contains at least one non-zero element of S and also does not contain every element of S we term I a non-zero proper ideal of S.

Definition 1. A mob is called simple if S does not contain a non-zero proper ideal.

Definition 2. An idempotent e of S is primitive if $f = f^2 \in eSe$ implies $f = 0$ or $f = e$.

Definition 3. A mob S is completely simple if S is simple and contains a non-zero primitive idempotent.

The zero semigroup of order $2, O_2$ clearly contains no non-zero proper ideals. Since it is not easy to classify this semigroup with the other simple semigroups, we shall assume in the following that $S \neq O_2$.

Lemma 1. A necessary and sufficient condition for a mob S to be simple is that $SxS = S$ for all non-zero x of S.

Proof. The condition is sufficient, since if I is a non-zero proper ideal of S and if $x \neq 0$, $x \notin I$ we have $SxS \subset I$, which contradicts $SxS = S$. Suppose now that S is simple and that the condition is not satisfied. Then there exists an element $x \neq 0$ such that $SxS = 0$, since SxS is an ideal of S. Let X be the set of all such x. Then clearly $XS \subset X$ and $SX \subset X$. Since X contains $x \neq 0$, $X = S$. $S^3 = SXS = 0$, so $S^2 = 0$.
But then for any $a \neq 0$, $a \in S$, $\{0,a\}$ is a proper ideal of S which is a contradiction.

Corollary. If $K \neq \emptyset$ is the kernel of the mob S, then K is a simple mob, for since K is the minimal ideal $KaK = K$ for all $a \in K$. Hence K is simple.

Lemma 2. If S is simple and e is an idempotent of S, then eSe is simple.
Proof. If $eSe = 0$, then trivial. Suppose $eSe \neq 0$, and let exe be any non-zero element of eSe. Then since S is simple $SexeS = S$.
Hence $eSe\, exe\, eSe = eSexeSe = eSe$, and lemma 1 implies that eSe is simple.

Lemma 3. If S is simple and e is a primitive idempotent then eSe is either a group or a group with zero.

Proof. Since eSe is simple there exists non-zero elements $a_x, b_x \in eSe$ such that $a_x x b_x = e$ for any $x \neq 0$. Then $xb_x a_x$ and $b_x a_x x$ are non-zero idempotents in eSe. Hence since e is primitive $xb_x a_x = e = b_x a_x x$. This implies that $eSe \setminus \{0\}$ is a group.

Theorem 1. If S is completely simple, then all idempotents of S are primitive.

Proof. Let e,f be two non-zero idempotents of S with e primitive. Since S is simple there exist elements $a, a' \in S$ such that $aea' = f$. Then if $b = fae$ and $b' = ea'f$ we have $bb' = f$, $fb = be = b$ and $b'f = eb' = b'$. Furthermore $b'b$ is an idempotent of S satisfying $eb'b = b'be = b'b$. Hence since e is primitive $b'b = e$.
Now the correspondence $x \to x'$, where $x \in b'Sb$ and $x' \in fSf$, which is defined by the equivalent relations

$x = b'x'b$ and $x' = bxb'$ is an isomorphism between b'Sb and fSf.
But since $b'Sb = b'fSfb = b'bb'Sbb'b = eb'Sbe \subset eSe$ and $eSe = b'bSb'b \subset b'Sb$, we have $b'Sb = eSe$. Hence $fSf \cong eSe$, and fSf is a group or a group with zero, which implies that f is a primitive idempotent.

Corollary. A completely simple semigroup S with the identity u is either a group or a group with zero. For by theorem 1, u is a primitive idempotent, and hence lemma 3 implies that $uSu = S$ is a group or a group with zero.

Theorem 2. A compact simple semigroup S is completely simple.

Proof. Let $a \neq 0$ be any element of S. Then since S is simple there exist b,c such that $bac = a$. Then $b^n a c^n = a \quad n=1,2,\dots$.
Let e be the identity of the group $D = \bigcap_{n=1}^{\infty} \overline{\{b^i \mid i \geq n\}}$ (see § 1 theorem 4).
Then by § 1 lemma 2 there exists $c' \in \overline{\{c^i \mid i=1,\dots\}}$ such that $eac'=a$. Hence $ea = a$, which implies $e \neq 0$.
Now let $f \neq 0$ be any idempotent in eSe. Since eSe is simple and compact, we can again apply § 1, th.4 and lemma 2. Hence there is an idempotent $g \in eSe$ and an element $g' \in eSe$ such that $gfg' = e$.
Since e is the identity of eSe, we have $g = ge = ggfg' = gfg' = e$ and $fg' = efg' = gfg' = e$. Henceforth $f = fe = ffg' = fg' = e$.
Thus e is the only idempotent $\neq 0$ contained in eSe, and e must be primitive.

Let I be an ideal of the mob S. Then the Rees quotient S/I is the semigroup which consists of the set S-I (the complement of I) with the relative topology, together with a single isolated element 0.
The multiplication in S/I is defined in the following way:

$a . b = ab$, if $a,b,ab \in S-I$.

$a . b = 0$ if $ab \in I$.

$a . b = 0$ if $a=0$ or $b=0$.

If I is a closed compact ideal of S, we define S/I in the following way. S/I is the space which we get from S by identifying I to a single point 0, with the quotient topology. Multiplication in S /I is defined in the same way as before.

Theorem 3. Let J be a maximal proper ideal of the compact mob S. Then S/J is either the zero semigroup of order 2, or else is completely simple.

Proof. If S-J consists of a single element x, then $S/J = \{0,x\}$, where $x^2 = x$, or $x^2 = 0$.
If $x^2 = 0$, then $S/J \simeq 0_2$. If $x^2 = x$, then S is completely simple, since S is simple and x is a non-zero primitive idempotent.
Now assume that S-J contains more than one element. Let A be an ideal of $S \setminus J$.
Then $A - \{0\} \cup J$ is an ideal of S containing J. Since J is maximal $A = 0$, or $A = S \setminus J$.
Hence S/J contains no non-zero proper ideal and S/J is simple.
Furthermore the compactness of S implies that J is open, and hence S/J is compact. Then by theorem 2 S/J is completely simple.

Lemma 4. Let S be a semigroup without zero having at least one minimal left ideal L. Then S is the sum of its minimal left ideals if and only if S is simple.

Proof. Let S be simple. According to § 2, th.2., the sum of all the minimal left ideals of S is a two-sided ideal I.
Hence I = S, since $I \subset S$ would be contrary to the simplicity of S.
Conversely, let S be the sum of its minimal left ideals $S = \sum_{\alpha} L_{\alpha}$.
Then again by § 2, th.2., S is its own minimal ideal, hence S is simple.

<u>Theorem 4.</u> Let S be a simple semigroup without zero, having at least one minimal left and one minimal right ideal. Then S is the class sum of disjoint isomorphic groups.

<u>Proof.</u> According to lemma 4 S = K, hence by § 2, th.3., $S = \bigcup \{ H(e) \mid e \in E \}$, where any pair $H(e_1)$, $H(e_2)$ are topologically isomorphic and $H(e_1) \cap H(e_2) = \emptyset$.

<u>Definition.</u> A mob is called left (right) simple if S does not contain a non-zero proper left (right) ideal.

Just as in lemma 1 we can prove that a necessary and sufficient condition for a mob S to be left (right) simple is that Sa = S (aS = S) for all non-zero $a \in S$. Furthermore it is clear that every left or right simple mob is simple.

<u>Lemma 5.</u> If $A \subseteq S$, S compact, and A a left (right) simple semigroup, then $\overline{A}$ is also a left (right) simple mob.

<u>Proof.</u> $\overline{A}$ is a subsemigroup of S (§ 1 Lemma 1), hence $\overline{A}x \subset \overline{A}$ for all $x \in \overline{A}$.

Now let A be left simple and suppose for $x \in \overline{A}$ $\overline{A}x \neq \overline{A}$. Then there exist $y \in \overline{A}$, $y \notin \overline{A}x$, and hence a neighbourhood V of x such that $y \notin \overline{A}V$. Since $x \in \overline{A}$, there is an element $a \in A \cap V$ and thus $y \notin \overline{A}a$, which is a contradiction. A similar argument applies to right simple mobs.

<u>Theorem 6.</u> Let e be an idempotent of the compact mob S without zero, then these are equivalent.

1) e is primitive.
2) Se is a minimal left ideal.
3) SeS is the minimal ideal.
4) eSe is a maximal subgroup.
5) each idempotent of SeS is primitive.

<u>Proof.</u> (1) → (2). If Se is not minimal then there exists an idempotent f with $Sf \subset Se$ and Sf a minimal left ideal (§ 2,th.3 and 4). Hence fe = f. Since (ef)(ef) = e(fe)f = eff = ef, ef is an idempotent $\in$ eSe. This implies that ef = e. Hence $e \in Sf \Rightarrow Se \subset Sf$ and Se is minimal.

(2) → (3) → (4) → (2) § 2, th.6.

(2) → (1). Let f be an idempotent with fe = ef = f. Then $f \in Se$, hence $Sf \subset Se$, and since Se is minimal Sf = Se. It now follows that e = ef = f.

(5) → (1) trivial.

(1) $\rightarrow$ (5). Let $f \in SeS$. Then since $SeS = K$, we have $S f S = K$ and thus f primitive ((3) $\rightarrow$ (1)).

Remark. Hence in a compact mob without zero, idempotents are primitive if and only if they are contained in K.

§ 4. Maximal ideals

We have seen in § 3, th.1., that if S is a compact mob which contains properly a (left, right) ideal, then it contains a maximal proper (left, right) ideal J and J is open.

Theorem 1. Let S be compact and suppose E is contained in a maximal proper ideal J, then $S^2 \subset J$.

Proof. Let $a \in S \setminus J$, then $SaS \subset J$ or $SaS \cup J = S$ since J is maximal. If $SaS \cup J = S$, then there exist $x, y \in S$ such that $xay = a$. Hence $x^n a y^n = a$, $n=1,2,\ldots$, and there is an idempotent $e \in \Gamma(x)$ and an element $y' \in \Gamma(y)$ such that $a = eay'$. This implies that $a \in Jay' \subset J$ is a contradiction.
If $SaS \subset J$ for all $a \in S \setminus J$, then $S^3 \subset J$.
Now let $S^2 \not\subset J$, then $S = S^2 \cup J \Rightarrow S^2 = S^3 \cup SJ \subset J$. This contradiction completes the proof.

Corollary. Let S be compact with $S^2 = S$, then $SES = S$. If SES is a proper subset of S, then since SES is an ideal it is contained in a maximal proper ideal J (§ 3,th.1). Hence by theorem 1 $S = S^2 \subset J$, a contradiction.

Theorem 2. Let S be compact with $S^2 = S$ and suppose S has a unique idempotent, then S is a topological group.

Proof. Let $e = e^2$, then $e \in K$ and K is a group. By the corollary $S = SeS = K$, completing the proof.

Definition. A mob S has the (left, right) maximal property if there exists a maximal proper (left, right) ideal (L^*, R^*) J^* containing every (left, right) ideal of S different from S.

Lemma 1. Let S be a mob and A a compact part of S. If $A \subset At$ with $\Gamma(t)$ compact, then $A = At = Ae \quad e \in \Gamma(t)$.

Proof. $A \subset At \subset At^2 \subset \ldots$
Suppose now $At^k \not\subset Ae \quad e \in \Gamma(t)$. Then there is an $a \in A$ with $at^k \notin Ae$, and there exist a neighbourhood W of e such that $at^k \notin A.W(e)$.
But since e is a clusterpoint of $\{t^n\}_{n=1}^{\infty}$, there is a $k_o \geq k$ with $t^{k_o} \in W(e)$.
Hence $at^k \notin At^{k_o}$, which is a contradiction.
We now have $A \subset At \subset Ae$, where $e^2 = e$: therefore $A = Ae$ and $A = At = Ae$.

<u>Lemma 2.</u> Let S have a right unit element e and at least one proper left ideal. Then S has the left maximal property.

<u>Proof.</u> Let L^* be the union of all proper left ideals. Then $L^* \neq \emptyset$, L^* is a left ideal and $e \notin L^*$.
For if $e \in L^*$, then $e \in L$ for some proper left ideal. But since e is a right unit $S = Se \subset L$, a contradiction.
Therefore $L^* \neq S$, and it is obvious that L^* is the maximal left ideal of S.
We remark that lemma 2 holds if "right" is replaced by "left" and vice versa.
Also a similar argument shows that if S has a left or right unit and at least one proper ideal, then J^* exists.
From the proof of the lemma it follows that if S has a left unit then R^* also exists and $J^* \subseteq R^*$; if S has a right unit, then $J^* \subseteq L^*$ and if S has a unit, then $J^* \subseteq L^* \cap R^*$.

<u>Theorem 3.</u> Let S be a compact mob. In this case if L^* exists, then there exists also J^* and we have $L^* = J^*$.
(The theorem also holds if L^* is replaced by R^*).

<u>Proof.</u> Since for every $s \in S$ L^*s is a left ideal of S, we have $L^*s \subset L^*$ or $L^*s = S$.
Lemma 1 implies that if $L^* \subset L^*s = S$, then $L^* = L^*s = S$, a contradiction.
Hence $L^*s \subset L^*$. But then it follows that $L^*S \subset L^*$. Hence L^* is an ideal of S, which must be J^*, since every proper ideal of S is a proper left ideal of S and is contained in L^*.
From theorem 3 it follows that if S is compact and L^* and R^* exist, then J^* also exists and $J^* = L^* = R^*$.

<u>Theorem 4.</u> Let S be a compact mob and let P be the set of those elements $a \in S$, satisfying $aS = S$. Then P is closed and if $a \in P$ and $ax = ay$, then $x = y$. Further $xy \in P$ implies $x, y \in P$ and $P = \bigcup \{H(e) \mid e \in E \cap P\}$. All $H(e)$ $e \in E \cap P$ are isomorphic.

<u>Proof.</u>

a) To show that P is closed take $x \in S \setminus P$ and $y \in S \setminus xS$. Then $xS \subset S \setminus y$, and since S is compact we can find an open set U, with $x \in U$ and such that $US \subset S \setminus y$. Then $x \in U \subset S \setminus P$.

b) Suppose now that $ax = ay$. $x \neq y$.

If $a \in P$, then $S = aS = a^2S = \ldots$ and by lemma 1, $S = eS$, $e \in \Gamma(a)$. Furthermore e is a left unit for S. From $ex = x$ and $ey = y$ we infer the existence of an open set U including e such that $Ux \cap Uy = \emptyset$. Since $e \in \Gamma(a)$, we know that some $a^n \in U$. But since $a^n x = a^n y$ we must have $x = y$.

c) If $xy \in P$, then $xyS = S$. Let $yS = A$.

Then lemma 1 implies that since $A \subset xA$, $A = xA$. Hency $yS = xyS = S$ and $y \in P$. But then since $xyS = S = xS$, $x \in P$.

d) Now let $a \in P$, then $aS = S$ and there exists an $f \in S$ with $af = a$.

(c) implies that $f \in P$ and (b) implies that f is unique. Since $af^2 = af = a$, we have $f^2 = f$ and f is a left unit for S. In the same way there exists a unique $a^{-1} \in P$ such that $aa^{-1} = f$. Since $aa^{-1}a = fa = a = af$; we have $a^{-1}a = f$.

Hence $a \in H(f)$, and (c) implies that $H(f) \in P$.

Thus $P = \bigcup \{ H(e) \mid e \in E \cap P \}$.

e) Now let $e,f \in E \cap P$. Then we shall prove that the map φ: $H(e) \to H(f)$ $\varphi(x) = xf$ is a topological isomorphism. It is clear that $xf \in P$.

Suppose now $xf \in H(g)$ $g \in E \cap P$, and let x^* be the inverse of xf in $H(g)$.

Then $x^*xf = g$, hence $gf = g$. But since $g \in E \cap P$ g is a left unit and $gf = f = g$. Thus $xf \in H(f)$.

(b) implies that φ is one to one, and it is obvious that φ is onto, since $\varphi(be) = bf = b$ for $b \in H(f)$.

We can also easily verify that φ is a homomorphism. Since $H(e)$ and $H(f)$ are both compact, it follows that φ is topological.

Theorem 5. Let S be compact and let $S \neq P \neq \emptyset$. Then $S \setminus P$ is the maximal proper ideal J^* of S.

Proof. (c) of theorem 4 implies that $S \setminus P$ is an ideal of S. Since P is a compact submob, it includes an idempotent which must be a left unit of S. Hence by the remark to lemma 2 J^* exists and $S \setminus P \subset J^*$. If $S \setminus P \neq J^*$, then there is $a \in P \cap J^*$. Then $S = aS \subset J^*S \subset J^*$. Therefore $S \setminus P = J^*$.

Corollary. If S is compact with unit u and if S is not a group, then $J^* = S \setminus H(u)$.

Proof. Since S = uS we have $H(u) \subset P$. Now let $e \in E \cap P$, then e is a left identity of S and hence eu = u = e. Therefore theorem 4 (d) implies that P = H(u). Hence by theorem 5: $S \setminus P = S \setminus H(u) = J^*$.

Theorem 6. Let S be compact and suppose R^* exists. If

a) $S-R^*$ has more than one element, or

b) S is connected,

then $S - R^*$ is a right simple closed semigroup and is the sum of disjoint isomorphic closed topological groups.

Proof.

a) Let $a \in S - R^*$. Then since $aS \cup \{a\}$ is a right ideal of S, not included in R^*, we have $aS \cup \{a\} = S$.
Hence $aS = R^*$ or aS = S. If $S - R^*$ has more than one element, then aS cannot be equal to R^*, hence aS = S.

b) Now let S be connected and let $S = aS \cup \{a\}$ $a \in S - R^*$. Then since both aS and $\{a\}$ are closed, we have by the connectedness of S that $a \in aS$, hence aS = S.
So we have in both cases for $a \in S - R^*$, aS = S.
Moreover it is clear that if $x \in S$, with xS = S, then $x \in S - R^*$ and thus $S - R^* = P$. From theorem 4 it follows that $S - R^*$ is the sum of disjoint isomorphic closed topological groups and that $S - R^*$ is closed.
Since P is a semigroup, we know that $aP \subset P$ for $a \in P$. Next let $b \in P$, then since aS = S, there is $b' \in S$, such that ab' = b. Theorem 4 (c) implies that $b' \in P$.
Hence we have shown that for all $a \in P$ aP = P and that $P = S - R^*$ is right simple.

Corollary. Let S be a compact connected mob with R^*. Then S contains at least one left unit element.

Proof. Since $S - R^* \neq \emptyset$ and $S - R^* = P$, we know that S contains an idempotent $e \in P$ (P a compact mob). But then e is a left unit, since every idempotent of P is a left unit of S.

Theorem 7. The necessary and sufficient condition that a connected compact semigroup S contains R^* is

a) S has at least one left unit element and

b) S is not right simple.

Proof. The necessity of the condition follows from the definition of R^* and the above corollary.

That the condition is sufficient follows from lemma 2.

Theorem 8. Let S be a compact mob and suppose that $S - L^*$ and $S - R^*$ have more than one element. Then

a) S has a unit u

b) $L^* = R^* = J^*$

c) $S = L^* \cup H$, $L^* \cap H = \emptyset$ H is a closed topological group with unit u.

Proof.

a) According to th.7 S has a left unit e_l and a right unit e_r. Since $e_l e_r = e_l = e_r$, e_l is a unit.

b) follows from th.3.

c) Theorem 6 implies that $S - R^* = S - L^* = H$ is a right and a left simple closed semigroup.

Hence $\mathcal{L} H = H\mathcal{L} = H$ for every $h \in H$. Hence H is a group, and it follows from the theorem that H is a closed topological group.

Theorem 9. Let S be a connected compact mob, having at least one left unit and suppose S is not right simple. Then every subgroup H(e), with e a left unit lies in the boundary of the maximal right ideal R^*.

Proof. Since $\overline{R^*}$ is also a right ideal of S, we have $\overline{R^*} = R^*$ or $\overline{R^*} = S$. If $\overline{R^*} = R^*$, then $S - R^*$ is open and $S - R^*$ is closed by th.6. Since S is connected, this is a contradiction.

Hence $\overline{R^*} = S$ and $S - R^* = \{\bigcup H(e) \quad e \text{ left unit}\} =$

$S \cap \overline{S - R^*} = \overline{R^*} \cap \overline{S - R^*}$ = boundary R^*.

§ 5. Prime ideals

Definition. A right [left or two-sided] ideal P of S is said to be prime if $A.B \subset P$ implies that $A \subset P$ or $B \subset P$, A and B being ideals of S. An ideal P^* is completely prime if $ab \in P^*$ implies that $a \in P^*$ or $b \in P^*$, $a, b \in S$.

An ideal which is completely prime is prime, but the converse is not generally true.

In the case of commutative mobs however, this concepts coincide. For let P be a prime ideal in a commutative mob and let $ab \in P$. Then $(a \cup aS)(b \cup bS) = ab \cup abS \subset P$, and hence $a \cup aS \subset P$ or $b \cup bS \subset P$. $\Rightarrow$ $a \in P$ or $b \in P$.

Lemma 1. If P is a left ideal of S, then the following conditions are equivalent.

1^o) P is a prime left ideal

2^o) If $aSbS \subset P$ then $a \in P$ or $b \in P$

3^o) If $R(a)\ R(b) \subset P$ then $a \in P$ or $b \in P$

4^o) If R_1, R_2 are right ideals of S such that $R_1R_2 \subset P$ then $R_1 \subset P$ or $R_2 \subset P$.

Proof.

(1) $\rightarrow$ (2): Let $aSbS \subset P$.

Then $R(a)^2R(b)^2 \subset aSbS \subset P$. Hence $J(a)^2J(b)^2 = (R(a)^2 \cup SR(a)^2)(R(b)^2 \cup SR(b)^2) = R(a)^2R(b)^2 \cup SR(a)^2R(b)^2 \subset P$. Since $J(a)^2$ and $J(b)^2$ are ideals of S, we have $J(a)^2$ or $J(b)^2 \subset P$.

If $J(a)^2 \subset P$, then $J(a) \subset P$ and hence $a \in P$.

(2) $\rightarrow$ (3). If $R(a)\ R(b) \subset P$, then $aSbS \subset P$, hence $a \in P$ or $b \in P$.

(3) $\rightarrow$ (4). Let $R_1R_2 \subset P$, and suppose $a \in R_1 - P$, $b \in R_2 - P$. Since $R(a) \subset R_1$ and $R(b) \subset R_2$ we have $R(a)\ R(b) \subset P$, and thus $a \in P$ or $b \in P$, a contradiction. This implies that either $R_1 \subset P$ or $R_2 \subset P$,

(4) $\rightarrow$ (1): Trivial.

A similar proof shows that lemma 1 holds, if we replace right by left and vice versa.

Condition 2 then becomes: If $SaSb \subset P$ then $a \in P$ or $b \in P$.

For two-sided ideals we have

Lemma 1'. If P is an ideal of S, then the following conditions are equivalent.

1^o) P is a prime ideal.

2^o) If $aSb \subset P$ then $a \in P$ or $b \in P$.

3^o) If $J(a)\ J(b) \subset P$ then $a \in P$ or $b \in P$.

4^o) If R_1, R_2 are right ideals of S such that $R_1 R_2 \subset P$ then $R_1 \subset P$ or $R_2 \subset P$.

5^o) If L_1, L_2 are left ideals of S such that $L_1 L_2 \subset P$ then $L_1 \subset P$ or $L_2 \subset P$.

Theorem 1. Let S be a mob and suppose $E \neq \emptyset$ and let $e \in E$. Then each of $J_o(S-e)$, $R_o(S-e)$ and $L_o(S-e)$ is prime if it is not empty.

Proof. Suppose that $a \notin J_o(S-e)$ and $b \notin J_o(S-e)$. Then since $J_o(S-e)$ is maximal $e \in J(a)$ and $e \in J(b)$. This implies that $e \in J(a)\ J(b)$ and hence $J(a)\ J(b) \not\subset J_o(S-e)$.

By property (3) of lemma 1', this shows that $J_o(S-e)$ is a prime ideal. The statement for $R_o(S-e)$ and $L_o(S-e)$ can be proved in the same way.

If $E \neq \emptyset$, we can define a partial ordering in E as follows: for $e, f \in E$ $e \leq f$ if and only if $ef = fe = e$. It is clear that the relation $\leq$, thus defined is reflexive and antisymmetric.

Now let $e \leq f$ and $f \leq g$. Then $ef = fe = e$ and $fg = gf = f$. Hence $eg = (ef)g = e(fg) = ef = e$, and $ge = g(fe) = (gf)e = fe = e$. This implies that $e \leq g$, and the relation $\leq$ is transitive.

If S is a mob without zero, then the minimal elements of E are the primitive idempotents.

If S has a zero, then the non-zero primitive idempotents are the atoms of the partly ordered set E.

Furthermore, if S has a unit u, then u is the maximal element of E.

Lemma 2. Let P be an open prime right [left] ideal of a compact mob S. If A is a left [right] ideal of S which is not contained in P, then A has an idempotent e with $Se \not\subset P$.

Proof. Let P be an open prime right ideal and let $a \in A - P$. Then L(a) is a compact left ideal with $L(a) \subset A$. $L(a) \not\subset P$. Now let $L_1 \supset L_2 \supset \ldots$ be a linearly ordered sequence of compact left ideals with $L_i \subset A$ $L_i \not\subset P$, If $L = \bigcap_{i=1} L_i$, then because P is open and the L_i compact $L \not\subset P$.

Now using Zorn's lemma there exists a minimal member L of the set of all compact left ideals L_α with $L_\alpha \subset A$, $L_\alpha \not\subset P$. Now let $a \in L \setminus P$, and suppose $La \subset P$.

Then $(a \cup La)(a \cup La) \subset La \subset P$.

Hence by lemma 1 (4). $a \cup La \subset P$ a contradiction. Thus $La \not\subset P$.

Since $La \subset L$ and L is minimal $La = L$.

Furthermore $La^n = La.a^{n-1} = La^{n-1} = \ldots = L \not\subset P$.

Now let $e \in \Gamma(a) \subset L$, then $L = Le$ by § 4 lemma 1, and hence $Le \not\subset P$, thus $Se \not\subset P$.

<u>Corollary</u>: Let P be an open prime ideal of the compact mob S. If A is a right or left ideal of S not contained in P, then A - P contains a non-minimal idempotent.

<u>Proof.</u> Let A be a left ideal.

Then it follows from lemma 2 that there exists $e \in A$ and $a \in A - P$ with $a \in Sa = Se \not\subset P$.

Thus $ae = a$, and since P is an ideal, it would follow from $e \in P$, that $ae = a \in P$, a contradiction. Hence $e \in A - P$.

Furthermore it is clear that $e \notin K$, since $K \subset P$.

If S is a mob with zero, then § 3, th.6., implies that e is non-primitive and hence non-minimal.

If S has a zero, then since $K = 0$, we have $e \neq 0 \Longrightarrow e > 0$.

<u>Theorem 2.</u> If S is compact, then each open prime ideal $P \neq S$, has the form $J_o(S-e)$, e non-minimal. If conversely e is a non-minimal idempotent, then $J_o(S-e)$ is an open prime ideal.

<u>Proof.</u> Let P be an open prime ideal. Then we can find in the same way as in lemma 2, a minimal ideal $J_1 = J(e)$, $J \not\subset P$ $e \in S - P$.

Now let $P^* = J_o(S-e)$, then P^* is an open prime ideal, and $P \subset P^*$.

Again using lemma 2, if $P \neq P^*$, we can find an idempotent $f \in P^* \setminus P$ with $J = J(f) \not\subset P$. Since $e, f \notin P$, $J(e)\ J(f) = J_1 J_2 \not\subset P$.

Furthermore $J_1 J_2 \subset J_1$, and since J_1 is minimal $J_1 J_2 = J_1$.

Hence $J_1 = J_1 J_2 \subset J_2 \subset P^*$ a contradiction.